院試hub
Worked solutions

MEXT

Undergraduate

Mathematics A　Worked solutions

Embassy recommendation　2016–2019

12 major problems / 68 questions

Question text is not included. Use with the official papers.
Official public examination papers, 2016–2019

inshihub.com

Part 3 2016 / 3 / Scores and correlation

Official question PDF: physical page 4, counted from the first page.

This question combines missing-data reconstruction, a boxplot and a correlation interval. Retain the paired observations when finding covariance; sorting is useful for the boxplot but must not destroy the student pairing.

3(1) Missing entries, means and spread

Solution

The ten scores for A total $10 \cdot 5 = 50$. The nine supplied scores sum to 44, so the missing score is 6 and its total with the corresponding B score is $6 + 5 = 11$.

The B scores sum to 50, giving mean 5.0. The total-score mean is $5.0 + 5.0 = 10.0$. For B, the squared deviations from 5 sum to

$$(-2)^2 + 0^2 + 1^2 + (-1)^2 + 0^2 + 3^2 + 1^2 + 0^2 + 0^2 + (-2)^2 = 20.$$

Using the variance of these ten observations, $s_B^2 = 20/10 = 2$, so $s_B = \sqrt{2} = 1.414\ldots$, which rounds to 1.4.

Ans. [3-1] 6; [3-2] 11; [3-3] 5.0; [3-4] 10.0; [3-5] 1.4.

Explanation

Which denominator The spread describes the listed scores, using the average squared deviation with divisor 10. This is not the unbiased estimator with divisor 9.

Do not round too soon Compute variance from the data and take the square root at the end. The official key also accepts 1.3 for [3-5], but direct calculation from the displayed scores gives 1.4 to one decimal place.

Answer format matters The mean fields explicitly require one decimal place. Write 5.0 and 10.0, even though their numerical values are integers; the official marking notes distinguish these forms from 5 and 10.

3(2) Construct the boxplot

Solution

The sorted totals are

$$6,\ 6,\ 8,\ 9,\ 10,\ 11,\ 12,\ 12,\ 13,\ 13.$$

The median is $(10 + 11)/2 = 10.5$. Taking the median of each five-value half gives $Q_1 = 8$ and $Q_3 = 12$. The minimum and maximum are 6 and 13. These five values match option (d).

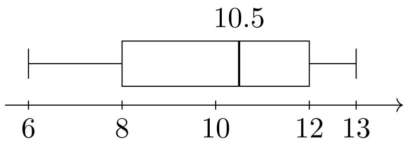
Ans. [3-6] (d).

Explanation

Quartiles from halves With ten observations, neither middle observation is dropped. The lower half is the first five values and the upper half is the last five; each has an unambiguous middle value.

Identify the decisive features Several choices share the same minimum and maximum. The lower edge of the box must be 8, and the median line must lie halfway between 10 and 11, rather than at 10.

An independent sketch The horizontal version below represents the same five-number summary. Orientation does not change the quartiles.



3(3) The correlation interval

Solution

Both subject means are 5. From the paired scores, the sums of squared deviations and cross-products are

$$\sum(A_i - 5)^2 = 44, \qquad \sum(B_i - 5)^2 = 20, \qquad \sum(A_i - 5)(B_i - 5) = 0.$$

For example, the nonzero cross-products are $4, 1, 1, -12, 2, 4$, whose sum is zero. Since neither subject has zero variance,

$$r = \frac{0}{\sqrt{44 \cdot 20}} = 0.$$

This lies in $-0.2 \leq r < 0.2$.

Ans. [3-7] (c).

Explanation

Use the original pairs Each product combines A and B for the same student. Sorting the two subjects independently would manufacture a positive association that the original data do not have.

A variance check The exact total-score variance is $6.4 = 4.4 + 2 + 2 \cdot 0$. Its square root is about 2.53, consistent with the displayed rounded standard deviation 2.5.

What zero means Zero correlation means the centered cross-products cancel. It does not say that the two lists of scores are identical, or prove independence. Here the task asks only for the interval containing the linear correlation.

Part 9 2018 / 3 / Cross-sections and volume scaling

Official question PDF: physical page 5, counted from the first page.

A circular cross-section provides the reference solid. Stretching it in two horizontal directions produces elliptical sections. Distinguish the radius from its square and multiply the two length scale factors for area.

3(1) Volume of the circular reference solid

Solution

At fixed height z, the inequality $x^2 + y^2 \le z^4$ describes a disk of radius z^2, because $z \ge 0$. Its area is πz^4. Therefore

$$V_B = \int_0^1 \pi z^4\, dz = \left[\frac{\pi z^5}{5}\right]_0^1 = \frac{\pi}{5}.$$

Ans. [3-1] $\frac{\pi}{5}$.

Explanation

Read the radius correctly The right side of a disk equation is the radius squared. Here $r^2 = z^4$, so $r = z^2$, not z^4.

Why slices give volume A thin horizontal slab has volume approximately its section area times its thickness dz. Integration adds those slabs over the height interval.

A comparison For $0 < z < 1$, $z^2 < z$, so this solid lies inside the cone of radius z and height one. Its volume $\pi/5$ is correspondingly less than $\pi/3$.

3(2) The two horizontal stretches

Solution

At positive height z, the section of A can be written

$$\frac{x^2}{(3z^2)^2} + \frac{y^2}{(2z^2)^2} \le 1.$$

It is an ellipse with semiaxes $3z^2$ and $2z^2$. Starting with the reference disk of radius z^2, multiply the x coordinate by 3 and the y coordinate by 2. At $z = 0$, both sections reduce to the origin.

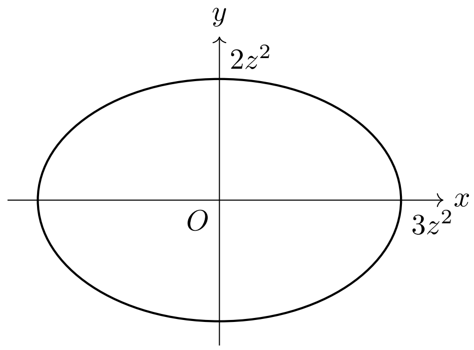
Ans. [3-2] 3; [3-3] 2.

Explanation

Squared coefficients versus length factors The coefficients 1/9 and 1/4 correspond to length factors $\sqrt{9} = 3$ and $\sqrt{4} = 2$. Using 9 and 4 would stretch lengths too far.

Check the coordinate transformation If (u, v, z) belongs to B, then $(3u, 2v, z)$ satisfies A's inequality, because $(3u)^2/9 + (2v)^2/4 = u^2 + v^2 \le z^4$.

A horizontal section The sketch labels the semiaxes at a fixed $z > 0$; it is a section, not a side view of the whole solid.



3(3) Ratio of the volumes

Solution

At every height the section area is multiplied by $3 \cdot 2 = 6$. The height coordinate is unchanged, so every slab volume has the same factor. Integrating preserves it:

$$V_A = \int_0^1 6\,(\pi z^4)\,dz = 6V_B.$$

Ans. [3-4] 6.

Explanation

Multiply independent scale factors Area changes by the product of the two perpendicular length factors. It is not multiplied by their sum, nor by the square of just one factor.

The third direction For a three-dimensional scaling, all three factors would be multiplied. Here the vertical factor is 1, giving $3 \cdot 2 \cdot 1 = 6$.

3(4) Volume of the elliptical solid

Solution

Combine the reference volume with the scale factor:

$$V_A = 6V_B = 6 \cdot \frac{\pi}{5} = \frac{6\pi}{5}.$$

Equivalently, integrate the ellipse area $\pi(3z^2)(2z^2) = 6\pi z^4$ directly from 0 to 1.

Ans. [3-5] $\frac{6\pi}{5}$.

Explanation

Two routes agree The scaling argument and direct section integral produce the same result. The former explains the factor six; the latter checks the height dependence.

A volume bound The enclosing elliptical cylinder has section area 6π and height one. The calculated volume is one fifth of that cylinder, consistent with sections that shrink to a point at the bottom.

Part 12 2019 / 3 / An integral-defined cubic

Official question PDF: physical page 6, counted from the first page.

The stationary points of the integral identify the zeros of its derivative. Keep the unknown leading coefficient until the normalization in the final part makes it cancel.

3(1) Parity of the integral

Solution

By the fundamental theorem of calculus, $F'(x) = f(x)$. The two distinct stationary points force

$$f(x) = c(x+2a)(x-2a) = c(x^2 - 4a^2), \qquad c \neq 0.$$

Integrating from zero gives

$$F(x) = c\left(\frac{x^3}{3} - 4a^2x\right).$$

Both terms are odd in x, so $F(-x) = -F(x)$.

Ans. [3-1] -1.

Explanation

Why there is no arbitrary constant The definition has lower limit zero, so $F(0) = 0$. An indefinite integral would contain a constant, but this definite-integral definition fixes it.

Use the correct stationary function The extreme points belong to F, so they are zeros of $F' = f$, not zeros of f'. Confusing these would impose the wrong quadratic.

Even integrand, odd integral The quadratic f is even. Integrating an even function from zero to x yields an odd function because reversing the integration direction changes the sign.

3(2) All solutions of the level equation

Solution

At $x = 2a$,

$$F(2a) = c\left(\frac{8a^3}{3} - 8a^3\right) = -\frac{16ca^3}{3}.$$

Thus $F(x) + F(2a) = 0$, after multiplication by $3/c$, becomes

$$x^3 - 12a^2x - 16a^3 = 0.$$

Factorization gives

$$(x+2a)^2(x-4a) = 0.$$

The distinct real solutions are $x = -2a$ and $x = 4a$.

Ans. [3-2] $-2a,\ 4a$.

Explanation

A predictable root Oddness gives $F(-2a) = -F(2a)$, so $-2a$ must solve the equation. This supplies a useful first factor before polynomial division.

Why the root is repeated The point $-2a$ is also stationary: $F'(-2a) = 0$. The horizontal level is tangent there, corresponding to the double factor $(x+2a)^2$.

List values, not multiplicities The double root is entered once. The factorization exhausts the cubic, so there are exactly two distinct solutions.

3(3) A normalized local maximum

Solution

The denominator is $F'(0) = f(0) = -4ca^2$, nonzero since $a > 0$ and $c \neq 0$. The normalized function is therefore

$$G(x) = \frac{F(x)}{F'(0)} = x - \frac{x^3}{12a^2}.$$

Differentiate:

$$G'(x) = 1 - \frac{x^2}{4a^2}, \qquad G''(x) = -\frac{x}{2a^2}.$$

At $x = 2a$, $G''(2a) = -1/a < 0$, so this is the local maximum. Its value is

$$G(2a) = 2a - \frac{8a^3}{12a^2} = \frac{4a}{3}.$$

Ans. [3-3] $\dfrac{4a}{3}$.

Explanation

Normalize before choosing the extremum The sign of c is unspecified, and the denominator may be negative. Dividing can reverse which extremum is a maximum. Working with the simplified G removes that uncertainty.

Maximum value versus location The local maximum occurs at $x = 2a$, but the requested answer is its height $4a/3$. Substituting the stationary input is the final step.

Local, not global The normalized cubic tends to positive infinity as x tends to negative infinity. It has no global maximum on the real line; $4a/3$ is specifically the local maximum.