院試hub
Worked solutions

MEXT

Specialized Training College

Mathematics Worked solutions

Embassy recommendation 2016–2019

12 major problems / 54 solution units

Question text is not included. Use with the official papers.
Official public examination papers, 2016–2019

inshihub.com

Part 3 2016: Recovering equations from graphs

Official question PDF: physical page 2, counted from its first page.

The three graphs supply different information: a vertex, a straight line, and two roots. Choose a form of the equation that uses the visible information immediately. The middle graph is allowed to have $a = 0$ even though the general expression contains an x^2 term.

3(1) Vertex form

Solution

The first graph has vertex $(-2, 1)$, so write

$$y = a(x + 2)^2 + 1.$$

Its vertical-axis intercept is 5, hence $5 = 4a + 1$ and $a = 1$. Expanding gives

$$y = (x + 2)^2 + 1 = x^2 + 4x + 5.$$

Ans. Fields $(a, b, c) = (1, 4, 5)$.

Explanation

Use the vertex directly The shift $x + 2$ puts the vertex at horizontal coordinate -2. Using $x - 2$ instead would reflect its position to the other side of the axis.

Check both observations At $x = -2$ the expression is 1; at $x = 0$ it is 5. The positive leading coefficient also matches the upward opening.

What is being tested The graph determines coefficients through structure, not through three arbitrarily chosen points. Vertex form reduces the calculation to one unknown.

3(2) A linear case

Solution

The graph is a straight line, so $a = 0$. Its intercepts give two points $(-3, 0)$ and $(0, 9)$, and its slope is

$$b = \frac{9 - 0}{0 - (-3)} = 3.$$

The vertical intercept is $c = 9$, so $y = 3x + 9$.

Ans. Fields $(a, b, c) = (0, 3, 9)$.

Explanation

Do not force a quadratic An expression $ax^2 + bx + c$ includes a line when $a = 0$. The drawing, not the presence of x^2 in a general formula, determines whether the quadratic coefficient vanishes.

Slope sign The line rises to the right, so its slope must be positive. Both numerator and denominator in the two-point formula must use the same point order.

Intercept check $3(-3) + 9 = 0$ and $3(0) + 9 = 9$. These are the two labeled intercepts.

3(3) Roots and a scale factor

Solution

The roots are -1 and 3, so write

$$y = a(x + 1)(x - 3).$$

At $x = 0$ the height is 6, giving $-3a = 6$ and $a = -2$. Thus

$$y = -2(x + 1)(x - 3) = -2x^2 + 4x + 6.$$

Ans. Fields $(a, b, c) = (-2, 4, 6)$.

Explanation

Roots do not fix the scale The two factors determine where the curve meets the axis, but the intercept is needed to determine a. Omitting it would leave infinitely many possible curves.

Check the shape The negative value $a = -2$ agrees with the downward opening. The symmetry axis is halfway between the roots, at $x = 1$.

Coefficient check Expanding $(x+1)(x-3)$ gives $x^2 - 2x - 3$. Multiplication by -2 changes the signs of both the linear and constant terms.

Part 11 2019: Two right triangles and a fifteen-degree angle

Official question PDF: physical page 3, counted from its first page.

The smaller right triangle fixes the scale of the larger isosceles one. Its 30° angle differs from the larger triangle's 45° angle by 15°. Keep the three radii and the perpendicular distance separate, and only then translate the exact results into the numbered fields.

2(1)–(4) Circumradii, inradius and perpendicular distance

Solution

Triangle ADC is right at C and has angle $D = 60°$. With $DC = 1$,

$$AD = \frac{1}{\cos 60°} = 2, \qquad AC = DC \tan 60° = \sqrt{3}.$$

Its circumradius is half its hypotenuse, so $R_{ADC} = 1$.
In triangle ABC, both legs have length $\sqrt{3}$. Its hypotenuse is $AB = \sqrt{6}$, and therefore $R_{ABC} = \sqrt{6}/2$. Its area is $3/2$ and its semiperimeter is $(2\sqrt{3} + \sqrt{6})/2$. The inradius is

$$r = \frac{3}{2\sqrt{3} + \sqrt{6}} = \frac{2\sqrt{3} - \sqrt{6}}{2}.$$

To find DH, choose $A = (0, 0)$, $C = (\sqrt{3}, 0)$, $B = (\sqrt{3}, \sqrt{3})$ and $D = (\sqrt{3}, 1)$. The line AB is $y = x$. A perpendicular through D has slope -1, so it meets AB at

$$H = \left(\frac{\sqrt{3}+1}{2}, \frac{\sqrt{3}+1}{2}\right).$$

Both coordinate differences from D to H have magnitude $(\sqrt{3} - 1)/2$. Thus

$$DH = \sqrt{2\left(\frac{\sqrt{3}-1}{2}\right)^2} = \frac{\sqrt{6} - \sqrt{2}}{2}.$$

Ans. (1) 1; (2) $\sqrt{6}/2$; (3) $(2\sqrt{3} - \sqrt{6})/2$; (4) $(\sqrt{6} - \sqrt{2})/2$.

The numbered fields in (3) are $2, 3, 6, 2$; those in (4) are $6, 2, 2$.

Explanation

Three different circles The first two radii are circumradii of different right triangles. The third is the inradius of ABC, which is smaller than either circumradius.

Check the inradius For a right triangle, $r = (AC + BC - AB)/2$. This gives $(2\sqrt{3} - \sqrt{6})/2$ directly and agrees with the area-over-semiperimeter calculation.

Check the foot of the perpendicular The point H lies on $y = x$. The displacement $D - H$ has opposite components, so it is perpendicular to the direction $(1, 1)$ of AB.

Why the square-root difference is positive Since $\sqrt{6} > \sqrt{2}$, the distance DH is positive.

Reversing the two radicals is not an equivalent distance.

2(5)–(6) Exact sine and cosine of the remaining angle

Solution

In triangle ADC, angle $DAC = 30°$. In the isosceles right triangle ABC, angle $BAC = 45°$. Since H lies on AB, the angle DAH is $45° - 30° = 15°$. Using the angle-difference identities,

$$\begin{aligned} \sin 15° &= \sin 45° \cos 30° - \cos 45° \sin 30° \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}, \\ \cos 15° &= \cos 45° \cos 30° + \sin 45° \sin 30° \\ &= \frac{\sqrt{6}+\sqrt{2}}{4}. \end{aligned}$$

Ans. (5) $(\sqrt{6}-\sqrt{2})/4$; (6) $(\sqrt{6}+\sqrt{2})/4$.

The numbered fields are $6, 2, 4$ in both parts; the signs between the radicals are already printed in the original.

Explanation

The cosine identity uses a plus sign For $\cos(u - v)$, the sine product is added. Copying the minus sign from the sine identity would make the cosine much too small.

Distance check Triangle ADH is right at H, so $\sin \angle DAH = DH/AD$. Using $DH = (\sqrt{6} - \sqrt{2})/2$ and $AD = 2$ gives the same sine.

Normalization check The squares add to

$$\frac{(\sqrt{6}-\sqrt{2})^2 + (\sqrt{6}+\sqrt{2})^2}{16} = \frac{16}{16} = 1.$$

Also $0 < 15° < 45°$, so the sine is smaller than the cosine, as the two expressions show.